

# Egyptian Fractions with odd denominators

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## Abstract

The number of solutions of the diophantine equation  $\sum_{i=1}^k \frac{1}{x_i} = 1$ , in particular when the  $x_i$  are distinct odd positive integers is investigated. The number of solutions  $S(k)$  in this case is, for odd  $k$ :

$$\exp\left(\exp\left(c_1 \frac{k}{\log k}\right)\right) \leq S(k) \leq \exp(\exp(c_2 k))$$

with some positive constants  $c_1$  and  $c_2$ . This improves upon an earlier lower bound of  $S(k) \geq \exp\left((1 + o(1)) \frac{\log 2}{2} k^2\right)$ .

## 1 Introduction

In this paper we study the number of solutions of the diophantine equation

$$(1.1) \quad \sum_{i=1}^k \frac{1}{x_i} = 1,$$

in particular, where the  $x_i$  have some restrictions, such as all  $x_i$  are distinct odd positive integers. Let us first review what is known for distinct positive

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integers, without further restriction: Let

$$\mathcal{X}_k = \{(x_1, x_2, \dots, x_k) : \sum_{i=1}^k \frac{1}{x_i} = 1, \quad 0 < x_1 < x_2 < \dots < x_k\}.$$

It is known that

$$(1.2) \quad \exp \left( \exp \left( ((\log 2)(\log 3) + o(1)) \frac{k}{\log k} \right) \right) \leq |\mathcal{X}_k| \leq c_0^{(\frac{5}{3} + \varepsilon) 2^{k-3}},$$

where  $c_0 = 1.264 \dots$  is  $\lim_{n \rightarrow \infty} u_n^{1/2^n}$ ,  $u_n = 1$ ,  $u_{n+1} = u_n(u_n + 1)$ .

The lower bound is due to Konyagin [12], the upper bound due to Brown- ing and Elsholtz [3]. Earlier results on the upper and lower bounds were due to Sándor [13] and Erdős, Graham and Straus (see [9], page 32).

The set of solutions has also been investigated with various restrictions on the variables  $x_i$ . A quite general and systematic investigation of expansions of  $\frac{a}{b}$  as a sum of unit fractions with restricted denominators is due to Graham [10]. Elsholtz, Heuberger, Prodinger [7] gave an asymptotic formula for the number of solutions of (1.1), with two main terms, when the  $x_i$  are (not necessarily distinct) powers of a fixed integer  $t$ .

Another prominent case is when all denominators  $x_i$  are odd. Sierpiński [16] proved that a nontrivial solution exists. It is known that for  $k = 9$  there are exactly 5 solutions, and for  $k = 11$ , there are exactly 379,118 solutions (see [15, 2]). Chen, Elsholtz and Jiang [4] showed that for odd denominators  $x_i$  the number of solutions of (1.1) is increasing with a lower bound of  $\sqrt{2}^{k^2(1+o(1))}$ . Other types of restrictions on the denominator have been studied, e.g. by Croot [5] and Martin [11]. The number of solutions of the equation  $\frac{m}{n} = \sum_{i=1}^k \frac{1}{x_i}$  have also been estimated by Elsholtz and Tao [8].

In this paper we take inspiration from the proof of Chen et al. [4] for odd denominators, and the proof of Konyagin [12] for lower bounds in the case of unrestricted  $x_i$ . As Konyagin's proof makes crucial use of ingenious identities, involving a lot of even numbers, it seems unclear whether one can generalize it to odd integers. Here is our main result:

**Theorem 1.1.** *Let  $s \geq 1$  and let  $\{p_1, \dots, p_s\}$  denote a set of primes, and let  $P = p_1 \cdots p_s$  be squarefree. Let  $k$  be sufficiently large. Moreover, if  $P$  is*

even, let  $k$  be odd. Let

$$\mathcal{X}_{k,P} = \{(x_1, x_2, \dots, x_k) : \sum_{i=1}^k \frac{1}{x_i} = 1, \text{ with distinct positive } x_i \equiv \pm 1 \pmod{P}\}.$$

There is some positive constant  $c(P)$  such that the following holds:

$$|\mathcal{X}_{k,P}| \geq \exp \left( \exp \left( c(P) \frac{k}{\log k} \right) \right).$$

The case  $P = 2$  is the case of odd denominators:

**Corollary 1.2.** *Let  $k$  be odd, and*

$$\mathcal{X}_{k,\text{odd}} = \{(x_1, x_2, \dots, x_k) : \sum_{i=1}^k \frac{1}{x_i} = 1, \text{ with odd distinct positive } x_i\}.$$

There is some positive constant  $c$  such that the following holds:

$$|\mathcal{X}_{k,\text{odd}}| \geq \exp \left( \exp \left( c \frac{k}{\log k} \right) \right).$$

For comparison, an upper bound of type  $\exp(\exp(c_2 k))$  follows from the unrestricted case, see (1.2).

## 2 Proof

**Lemma 2.1.** *Let  $P > 1$  be a squarefree integer. Let  $\omega(n)$  denote the number of distinct prime factors of  $n$ , and  $d(m)$  the number of divisors of  $n$ . The following holds:  $\omega(P^m - 1) \geq d(m) - 6$ .*

*Proof.* Due to a result of Bang, Zsigmondy, Birkhoff and Vandiver (see e.g. Schinzel [14]), it is known that for  $n > 6$  the values of  $P^n - 1$  have at least one *primitive* prime factor. (A prime factor of the sequence  $P^n - 1$  is primitive if it divides  $P^n - 1$ , but does not divide any  $P^m - 1$  with  $m < n$ .)

Let  $m = m_1 m_2$ . For each divisor  $m_1$  one has the factorization

$$P^m - 1 = (P^{m_1} - 1)(P^{m_1 m_2 - m_1} + P^{m_1 m_2 - 2m_1} + \dots + P^{m_1} + 1),$$

hence the number of prime factors of  $P^m - 1$  is at least the sum of the number of primitive prime factors of  $P^{m_1} - 1$ , for all possible divisors  $m_1$  of  $m$ .  $\square$

**Lemma 2.2.** *For  $X \geq 3$ , there exists a natural number  $m < X$  such that  $d(m) > \exp((\ln 2 + o(1))\frac{\ln X}{\ln \ln X})$  as  $X \rightarrow \infty$ .*

This follows from a theorem of Wigert [17], but can also be seen directly. Let  $P_r = \prod_{i=1}^r q_i$  be the product over the first primes, and choose  $m = P_r$ , if  $P_r \leq X < P_{r+1}$ . Then  $d(m) = 2^r = \exp((\ln 2 + o(1))\frac{\ln m}{\ln \ln m}) = \exp((\ln 2 + o(1))\frac{\ln X}{\ln \ln X})$ . Taking the first  $r$  odd primes, one can also find an odd number  $m$  of this type.

**Lemma 2.3.** *For every  $a, b, n_0 \in \mathbb{N}$  the following holds: every positive integer can be written as a finite sum of distinct fractions of the form  $\frac{1}{an+b}$ ,  $n \geq n_0$ .*

This result with  $n_0 = 0$  was originally proved by van Albada and van Lint [1]. The result for general  $n_0$  easily follows by using the progression  $a'n + b' = an + (an_0 + b), n \geq 0$ .

As an easy consequence we have:

**Lemma 2.4.** *There exist distinct positive integers*

$$l_1, \dots, l_{r_1}, m_1, \dots, m_{r_2}, n_1, \dots, n_{r_3},$$

*all larger than 1, in the residue class  $1 \bmod 3P(P^2-1)$  such that the following holds:*

$$\sum_{i=1}^{r_1} \frac{1}{l_i} = P - 2, \quad \sum_{i=1}^{r_2} \frac{1}{m_i} = 1, \quad \sum_{i=1}^{r_3} \frac{1}{n_i} = P,$$

If  $P = 2$ , then  $r_1 = 0$ , otherwise  $r_1, r_2, r_3 > 0$ . Moreover, it is clear that  $r_2 \equiv 1 \bmod P$ .

*Proof of Theorem.* The idea employed in [4] and [12] is to write 1 as a sum of fractions where one denominator has a large number of divisors, and to split this fraction recursively into several fractions, where (at least) one of these has again a large number of divisors.

Here we show that it is possible to have, for any given  $t \in \mathbb{N}$ , the fraction  $\frac{1}{P^t-1}$  as one of these fractions. Let us start with the trivial decomposition

$$1 = \frac{1}{P-1} + \frac{P-2}{P-1}.$$

In order to avoid that the denominator  $P - 1$  occurs more than once we use Lemma 2.4 to write the integer  $P - 2$  as a sum of distinct unit fractions, with  $l_i > 1$ :  $P - 2 = \sum_{i=1}^{r_1} \frac{1}{l_i}$ .

Next we observe that any fraction  $\frac{1}{P^n - 1}$  can be decomposed to obtain a sum of unit fractions containing a)  $\frac{1}{P^{2n} - 1}$  or b)  $\frac{1}{P^{n+1} - 1}$ .

$$(a) \quad \frac{1}{P^n - 1} = \frac{1}{P^n + 1} + \frac{1}{P^{2n} - 1} + \sum_{i=1}^{r_2} \frac{1}{(P^{2n} - 1)m_i}.$$

By Lemma 2.4

$$1 = \sum_{i=1}^{r_2} \frac{1}{m_i}, \quad m_i \equiv 1 \pmod{3P(P^2 - 1)}, m_i > 1 \text{ and distinct.}$$

Note that all occurring denominators are distinct, with the possible exception that  $P^n + 1 = P^{2n} - 1$  holds if  $P = 2, n = 1$ . In this case, one rewrites  $\frac{1}{P+1} = \frac{1}{3} = \sum_{i=1}^{r_2} \frac{1}{3m_i}$ . These denominators have not been used before, as the  $l_i$  or  $m_i$  are congruent to 1 mod 3, whereas the new denominators  $3m_i$  are not.

$$(b) \quad \frac{1}{P^n - 1} = \frac{1}{P^{n+1} - 1} + \frac{P - 1}{(P^n - 1)(P^{n+1} - 1)} + \frac{P - 1}{(P^{n+1} - 1)}.$$

Note that these three fractions are unit fractions, as the denominators are divisible by  $P - 1$ . These three fractions are distinct, unless  $n = 1$ . In this case the fraction  $\frac{1}{P^2 - 1}$  occurs twice and one of these is rewritten as  $\frac{1}{P^2 - 1} = \sum_{i=1}^{r_2} \frac{1}{(P^2 - 1)m_i}$ . These denominators have not been used before, as the previous denominators  $l_i$  and  $m_i$  were by construction congruent to 1 mod  $P^2 - 1$ . Also,  $P^n + 1, P^{2n} - 1, (P^{2n} - 1)m_i$  are new.

For constructing a solution with  $\frac{1}{P^t - 1}$  we write  $t$  in binary. The first binary digit is of course 1. For the positions  $i \geq 2$  we perform two different types of steps, corresponding to (a) and (b) above:

- 1) If the  $i$ -th leading position is a 0, then we take the “doubling” a).
- 2) If the  $i$ -th leading position is a 1, then we first take the doubling a), followed by an “addition” b),

For example, if  $t = 53 = 110101_2$  and starting from left to right:

$$\begin{array}{cccccccc}
i = 1 & | & 2 & | & 3 & | & 4 & | & 5 & | & 6 \\
1 & | & 1 & | & 0 & | & 1 & | & 0 & | & 1 \\
& | & a & | & b & | & a & | & b & | & a & | & a & | & b \\
n = 1 & | & 2 & | & 3 & | & 6 & | & 12 & | & 13 & | & 26 & | & 52 & | & 53
\end{array}$$

Generally, any integer  $t$  can be obtained in at most  $2^{\frac{\log t}{\log 2}}$  such steps a) or b). In other words, starting from  $n = 1$  we can obtain a decomposition

$$1 = \frac{1}{P^t - 1} + \sum_{i=1}^{k'-1} \frac{1}{x_i}$$

with  $k' = O(r_1 + r_2 \log t + r_3) = O_P(\log t)$  unit fractions. Observe that all denominators have been rearranged to be distinct.

We next come to the most crucial step, which determines the number of solutions:

**Lemma 2.5.** *Let  $\sum_{i=1}^{r_3} \frac{1}{n_i} = P$  (by Lemma 2.4).*

a) *For any divisor  $d | (P^t - 1)$  the following is an identity.*

$$\frac{1}{P^t - 1} = \frac{1}{P^t - 1 + Pd} + \sum_{i=1}^{r_3} \frac{1}{\frac{P^t - 1}{d} (P^t - 1 + Pd) n_i}.$$

b) *The number of divisors  $d | P^t - 1$  with  $d \equiv 1 \pmod{P}$  is at least  $2^{\frac{\omega(P^t - 1)}{P}}$ .*

c) *If  $d \equiv 1 \pmod{P}$ , then all denominators are  $\pm 1 \pmod{P}$ .*

Part a) and c) are easy to verify. For part b) observe: For any  $P$  prime factors  $p_k$ , being coprime to  $P$ , there is at least one subset of these primes, whose product is  $1 \pmod{P}$ . Indeed, the sequence  $a_1 = p_1, a_2 = p_1 p_2, \dots, a_P = \prod_{k=1}^P p_k$  must have two members  $a_i, a_j$ , say, which are equivalent modulo  $P$ . Then  $\frac{a_j}{a_i} = \prod_{k=i+1}^j p_k \equiv 1 \pmod{P}$ . Therefore, the number of divisors  $d \equiv 1 \pmod{P}$  is at least  $2^{\frac{\omega(P^t - 1)}{P}}$ . (Clearly, this argument can be refined (see e.g. [6]), but this would not improve our final result.) All solutions

produced in this way are distinct, as each solution has a unique denominator  $P^t - 1 + Pd$ . Moreover, as all these denominators are greater than  $P^t$ , and as in our application  $t$  will be chosen large, these new denominators are greater than those that have been used before.

We choose  $t$  as a product of the first primes. By Lemma 2.2 the number of divisors, and hence the number of solutions satisfies:

$$|\mathcal{X}_{k,P}| \geq 2^{\omega(P^t-1)/P} \geq 2^{(d(t)-6)/P} \geq 2^{\exp(\frac{\log 2 + o(1)}{P} \frac{\log t}{\log \log t})} \geq \exp(\exp(c(P)k/\log k)).$$

Recall that the number of fractions is  $k = O_P(\log t)$ .

Finally let us comment on the condition that  $k$  is odd, (see statement of the Theorem), when  $P$  is even. By multiplying equation (1.1) by its common denominator, and reducing modulo  $P$  it is clear that this condition is necessary. The condition is also sufficient as in view of step a) we can replace one fraction by  $r_2 + 2$  fractions. Again, by the same argument  $r_2 \equiv 1 \pmod{P}$ , so that effectively we replace one fraction by 3 fractions (modulo  $P$ ). Iterating this, we can reach any residue class modulo  $P$ , when  $P$  is odd, and the odd residue classes, when  $P$  is even. The number of extra fractions required is  $O(P r_2) = O_P(1)$ . This does not influence the overall result. In any case, the theorem is valid for sufficiently large  $k \geq k_P$ , with this necessary and sufficient congruence obstruction.

□

**Remark 2.6.** We have not worked out the constant  $c(P)$ . One may observe that  $c(P)$  might be as small as  $\frac{1}{r_2}$ . To estimate  $r_2$  one observes that  $\sum_{i=1}^{r_2} \frac{1}{i 3P(P^2-1)-1} \geq \sum_{i=1}^{r_2} \frac{1}{m_i} \approx \frac{\log r_2}{3P(P^2-1)} > P$  must hold. Hence  $r_2$  appears to be at least of exponential growth in  $P$ . Taking denominators  $x_i$  only coprime to  $P$ , but not necessarily restricted to  $x_i \equiv \pm 1 \pmod{3P(P^2-1)}$  would improve this constant  $c(P)$ .

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